AN APPROXIMATE SOLUTION OF THE BOUNDARY LAYER EQUATIONS WITH BLOWING PRESENT

A. M. Golovin and É. D. Sergievskii

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It is shown that the linearized equation of motion of a fluid flowing past a wedge with vertex angle $\pi/2$ and π, employed with blowing present over the whole boundary layer region, leads to results which agree well with those obtained earlier by a numerical method. The concentration and temperature fields are investigated for large and small Schmidt and Prandtl numbers.

The stationary two-dimensional flow of an incompressible material over a permeable surface with uniform blowing or suction present may be described by a system of laminar boundary layer equations, which, with constant physical properties and omitting energy dissipation and the work of the pressure forces, assume the following form [1, 2]:

$$\begin{gathered} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \\ \rho u\frac{\partial c}{\partial x}+\rho v\frac{\partial c}{\partial y}=D\frac{\partial^2 c}{\partial y^2}, \\ \rho u c_p\frac{\partial T}{\partial x}+\rho v c_p\frac{\partial T}{\partial y}=\varkappa\frac{\partial^2 T}{\partial y^2}, \end{gathered} \tag{1}$$

where x is the distance along the surface from a forward critical point of the body; y is the distance along the normal; u, v are the longitudinal and transverse components of the velocity, respectively; U is the speed at the exterior edge of the boundary layer; c is the concentration of the material blown in, defined as the ratio of the density of the material blown in to the density of the mixture; D is the diffusion coefficient of the binary mixture; T, ρ, $\varkappa$, c_p, and ν are, respectively, the temperature, density, thermal conductivity, heat capacity, and viscosity of the mixture.

The system of equations (1) is supplemented by the following boundary conditions:

$$\begin{gathered} u=0,\quad v=v_w,\quad c=c_w,\quad T=T_w \text{ for } y=0; \\ u=U,\quad c=c_\infty,\quad T=T_\infty \text{ for } y\to\infty. \end{gathered} \tag{2}$$

In the sequel we denote all quantities, which relate to the surface, with a subscript w, and quantities applicable to the domain far from the surface will be denoted with subscript ∞.

The concentration c_w and the blowing velocity v_w are connected through a relation which indicates the absence of the normal component of the main stream of the fluid at the surface:

$$v_w=-\frac{D}{1-c_w}\left(\frac{\partial c}{\partial y}\right)_w \tag{3}$$

The temperature of the surface T_w, subject to the condition that all the heat incident on the wall goes toward heating the blowing fluid to this temperature, is connected with the speed of blowing by the relation

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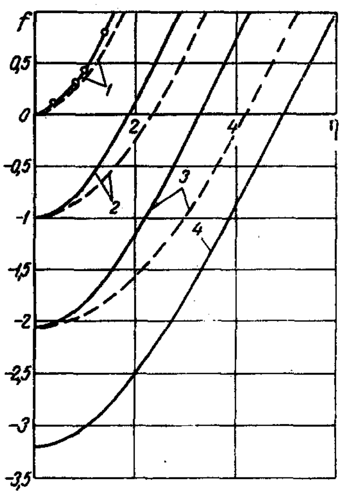

Fig. 1. Distribution of the stream function obtained for $\Lambda = 1$ (continuous curves) and $\Lambda = 0.5$ (dashed curves): Curve 1: $f_w = 0$, points denote numerical results from [1]; 2: $f_w = -1$; 3: $f_w = -2.03$; 4: $f_w = -3.19$.

$$v_w = \frac{\varkappa}{\rho c_p (T_w - T_0)} \left(\frac{\partial T}{\partial y}\right)_w, \tag{4}$$

where T_0 is the temperature of the inner surface of the porous wall.

Following [3, 4] we introduce new variables ξ and η, a stream function ψ, a dimensionless concentration C, and a dimensionless temperature θ

$$\xi = \frac{1}{U_\infty L}\int_0^x U(x)\,dx, \quad \eta = \frac{yU(x)}{U_\infty L}\left(\frac{\mathrm{Re}}{2\xi}\right)^{\frac{1}{2}}, \quad \left(\mathrm{Re} = \frac{U_\infty L}{\nu}\right),$$

$$\Psi = U_\infty L\left(\frac{2\xi}{\mathrm{Re}}\right)^{\frac{1}{2}} f(\xi, \eta), \quad \left(u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}\right),$$

$$C = \frac{c_w - c}{c_w - c_\infty}, \quad \theta = \frac{T - T_w}{T_\infty - T_w}.$$

Here L is a characteristic scale of length.

Further, we consider flows in which f, C, and θ depend only on the variable η. In this case the speed at the outer edge of the boundary layer varies in the following way: $U(x) \sim x^m$ (m = const). Such flows are realized for flow past a wedge of vertex angle $\pi\Lambda$.

The system of equations (1) then assumes the form:

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \Lambda\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0, \quad \left(\Lambda = \frac{2m}{m+1}\right),$$

$$\frac{d^2 C}{d\eta^2} + \mathrm{Sc}\, f\frac{dC}{d\eta} = 0 \quad \left(\mathrm{Sc} = \frac{\nu}{D}\right), \tag{5}$$

$$\frac{d^2\theta}{d\eta^2} + \mathrm{Pr}\, f\frac{d\theta}{d\eta} = 0 \quad \left(\mathrm{Pr} = \frac{\nu\rho c_p}{\varkappa}\right).$$

The boundary conditions (2) transform in the new variables to

$$\left(\frac{df}{d\eta}\right)_w = 0, \quad f_w = -\frac{v_w}{U}\left[\frac{2Ux}{(m+1)\nu}\right]^{\frac{1}{2}}, \; C_w = 0, \; \theta_w = 0 \text{ for } \eta = 0;$$

$$\left(\frac{df}{d\eta}\right)_\infty = 1, \quad C_\infty = 1, \quad \theta_\infty = 1 \quad \text{for} \quad \eta \to \infty. \tag{6}$$

The system of equations (5) with the boundary conditions (6) was solved numerically in [5-7]. The asymptotic solutions of this system, assuming blowing tending to infinity, were obtained in [3, 8-10]. We investigate below an approximate solution of the linearized system of boundary layer equations with blowing present.

1. When $\eta \gg 1$, as a consequence of the boundary condition (6) the stream function is equal to $f = \eta - \lambda$, where λ is an arbitrary constant. We seek a solution of the first of Eqs. (5) in the form

$$f = f_0 + f_1 \quad (f_0 = \eta - \lambda).$$

If $|f_1| \ll |\eta - \lambda|$, then following [6, 7], we may linearize this equation:

$$\frac{d^3 f_1}{d\eta^3} + (\eta - \lambda)\frac{d^2 f_1}{d\eta^2} - 2\Lambda\frac{df_1}{d\eta} = 0. \tag{7}$$

Equation (7) describes the flow rather precisely for large values of η. However the condition $|f_1| \ll |\eta - \lambda|$ is not satisfied for $\eta < \lambda$. Actually, Eq. (7) applies when the quadratic terms, neglected on the right side of the equations, are small. Further we shall show that when $\Lambda = 0.5$ and $\Lambda = 1$ a partial mutual compensation of the nonlinear terms takes place. Hence when $\Lambda = 0.5$ and $\Lambda = 1$ the solution of the linearized Eq. (7), satisfying the boundary conditions (6), turns out to be suitable for describing the velocity field in the boundary layer domain also.

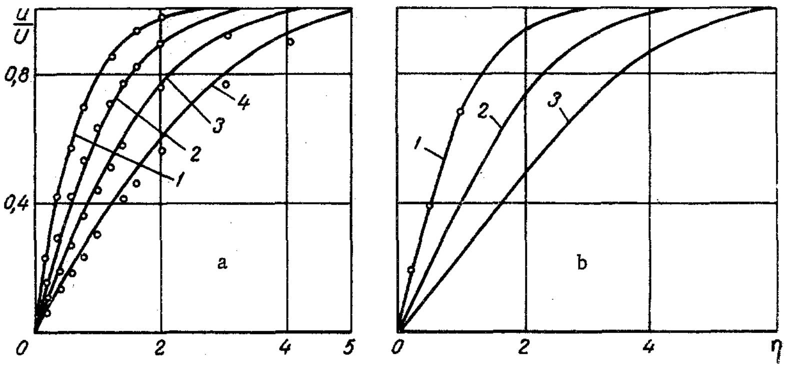

Fig. 2. Longitudinal velocity distribution obtained by the method of this paper for (a) $\Lambda = 1$ and (b) $\Lambda = 0.5$. Points plotted in Fig. 2a correspond to calculations from [5]; those plotted in Fig. 2b correspond to calculations from [6]. Curves 1, 2, 3, 4 correspond, respectively, to f_w values of 0, −1, −2.03, −3.19.

A solution of Eq. (7), satisfying the condition $df_1/d\eta \to 0$ for $\eta \to \infty$, may be expressed in terms of a parabolic cylinder function [11], an integral representation for which allows us to write the stream function in the form

$$f = \eta - \lambda + A \int_{\eta-\lambda}^{\infty} \int_{z}^{\infty} (y - z)^{2\Lambda} \exp\left(-\frac{y^2}{2}\right) dydz, \tag{8}$$

where

$$\frac{1}{A} = \int_{-\lambda}^{\infty} (y + \lambda)^{2\Lambda} \exp\left(-\frac{y^2}{2}\right) dy.$$

The boundary condition $f(0) = f_w$ enables us to find the constant λ

$$f_w + \lambda = A \int_{-\lambda}^{\infty} \int_{z}^{\infty} (y - z)^{2\Lambda} \exp\left(-\frac{y^2}{2}\right) dydz. \tag{9}$$

From the solution (8) it follows that

$$f = f_w + \frac{a\eta^2}{2} \quad \text{for} \quad \eta \ll \lambda, \tag{10}$$

where

$$\begin{gathered} a = \left(\frac{d^2 f}{d\eta^2}\right)_w = 2\Lambda A \int_{-\lambda}^{\infty} (y + \lambda)^{2\Lambda - 1} \exp\left(-\frac{y^2}{2}\right) dy, \\ f = \eta - \lambda + A \frac{\Gamma(1 + 2\Lambda)}{(\eta - \lambda)^{2\Lambda+2}} \exp\left[-\frac{(\eta - \lambda)^2}{2}\right] \quad \text{for} \quad \eta \gg \lambda, \end{gathered} \tag{11}$$

where

$$\Gamma(x + 1) = \int_0^{\infty} t^x \exp(-t)\, dt.$$

The expressions obtained simplify substantially for a large blowing speed ($\lambda \gg 1$). In this case the stream function (8) assumes the form

$$f = \eta - \lambda + \frac{\lambda}{2\Lambda + 1}\left(1 - \frac{\eta}{\lambda}\right)^{2\Lambda+1} \quad \text{for} \quad \lambda - \eta \gg 1, \tag{12}$$

$$f = \eta - \lambda \quad \text{for} \quad \eta - \lambda \gg 1. \tag{13}$$

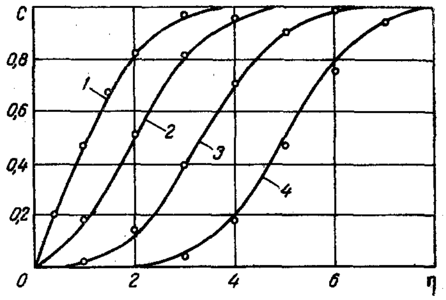

Fig. 3. Distribution of the relative concentration according to formula (26) for $\Lambda = 1$ (Sc = 0.7): curves 1, 2, 3, 4 correspond to f_W-values 0, −1, −2.03, −3.19, respectively; the points plotted refer to the numerical calculations from [5].

Here λ is determined from the equation

$$(2\Lambda + 1) f_w = -2\Lambda\lambda. \tag{14}$$

It is evident from this that the variation of the longitudinal component of the speed with distance along the normal has a point of inflection, for large blowing speeds, in the region $\eta \sim \lambda$ for wedge angles $\Lambda < 0.5$, which is evidently indicative of instability [1, 2].

It should be noted that for the case corresponding to $\Lambda = 1$ the Navier–Stokes equations, describing the plane-parallel flow close to the forward critical point of a cylindrical body, reduce to the first of Eqs. (5). We give below the results of our calculations of the velocity field for this case $\Lambda = 1$, and also for the case of flow past a wedge with a right angle at its vertex ($\Lambda = 0.5$).

For the case $\Lambda = 1$

$$f = \eta - \lambda + A_1 \left\{ \frac{2}{3} \left[1 + \frac{1}{2} (\eta - \lambda)^2 \right] \exp \left[-\frac{1}{2} (\eta - \lambda)^2 \right] - \frac{1}{2} \sqrt{2\pi} (\eta - \lambda) [1 + \Phi (\lambda - \eta)] - \frac{1}{6} \sqrt{2\pi} (\eta - \lambda)^3 [1 + \Phi (\lambda - \eta)] \right\}, \tag{15}$$

$$\frac{df}{d\eta} = 1 - A_1 \left\{ \frac{1}{2} \sqrt{2\pi} [1 + \Phi (\lambda - \eta)] [1 + (\eta - \lambda)^2] - (\eta - \lambda) \exp \left[-\frac{(\eta - \lambda)^2}{2} \right] \right\}, \tag{16}$$

where

$$\frac{1}{A_1} = \frac{1}{2} \sqrt{2\pi} [1 + \Phi (\lambda)] (1 + \lambda^2) + \lambda \exp \left(-\frac{1}{2} \lambda^2 \right),$$

$$\Phi (x) = \frac{2}{\sqrt{2\pi}} \int_0^x \exp \left(-\frac{t^2}{2} \right) dt.$$

The quantity λ is determined from Eq. (9), which assumes the form

$$\frac{1}{3} \sqrt{2\pi} [1 + \Phi (\lambda)] \lambda^3 + \frac{1}{2} \sqrt{2\pi} [1 + \Phi (\lambda)] (1 + \lambda^2) f_w + \left(\frac{2}{3} \lambda^2 - \frac{2}{3} + \lambda f_w \right) \exp \left(-\frac{1}{2} \lambda^2 \right) = 0. \tag{17}$$

Similar expressions are obtained for the case $\Lambda = 0.5$:

$$f = \eta - \lambda + A_2 \left\{ \frac{1}{4} \sqrt{2\pi} [1 + \Phi (\lambda - \eta)] [1 + (\eta - \lambda)^2] - \frac{1}{2} \exp \left[-\frac{1}{2} (\eta - \lambda)^2 \right] \right\}, \tag{18}$$

$$\frac{df}{d\eta} = 1 - A_2 \left\{ \frac{1}{2} \sqrt{2\pi} [1 + \Phi (\lambda - \eta)] (\lambda - \eta) + \exp \left[-\frac{1}{2} (\eta - \lambda)^2 \right] \right\}, \tag{19}$$

where

$$\frac{1}{A_2} = \frac{1}{2} \sqrt{2\pi} [1 + \Phi (\lambda)] + \exp \left(-\frac{1}{2} \lambda^2 \right).$$

The equation for determining λ is

$$\frac{\sqrt{2\pi}}{4} [1 + \Phi (\lambda)] (1 - \lambda^2) - \frac{\sqrt{2\pi}}{2} [1 + \Phi (\lambda)] \lambda f_w - \left(\frac{\lambda}{2} + f_w \right) \exp \left(-\frac{\lambda^2}{2} \right) = 0. \tag{20}$$

The values of the stream function (see Fig. 1) were calculated from formulas (15) (continuous curves) and (18) (dashed curves). The points plotted in the figure indicate results from previous calculations [1], made for the case without blowing. Values of the longitudinal components of velocity, calculated from formulas (16) and (19), are given in Fig. 2. The points plotted there denote the results of numerical calculations, for the same blowing quantities, using the data of [5] for $\Lambda = 1$ and the data of [6] for $\Lambda = 0.5$. Agreement with the results of the numerical calculations is satisfactory. The maximum error does not exceed 10-15%

for the blowing range considered. The quantity λ, defined by Eqs. (17) and (20), enables us to calculate the displacement thickness (see [2]), equal to

$$\delta^* = \lambda \sqrt{\Lambda \frac{\nu x}{mU}}. \tag{21}$$

The divergence of the results obtained in calculating λ [2] from those obtained in this paper without blowing amount to 1% for $\Lambda = 1$ and 3.7% for $\Lambda = 0.5$.

The domain of applicability of Eq. (7) is determined by the following inequality:

$$\max\left\{\left|(\eta-\lambda)\frac{d^2f_1}{d\eta^2}\right|,\quad 2\Lambda\left|\frac{df_1}{d\eta}\right|\right\} \gg \left|f_1\frac{d^2f_1}{d\eta^2} - \Lambda\left(\frac{df_1}{d\eta}\right)^2\right|. \tag{22}$$

The correction to the stream function f_1 turns out to be monotonic for $\Lambda \geq 0.5$. The maximum error occurs at $\eta = 0$. In this case, to satisfy inequality (22) it is sufficient to require that

$$\max\left\{\left|\lambda\left(\frac{d^2f}{d\eta^2}\right)_w\right|,\quad 2\Lambda\right\} \gg \left|(f_w+\lambda)\left(\frac{d^2f}{d\eta^2}\right)_w - \Lambda\right|. \tag{23}$$

Using the results of the numerical calculation it may be verified that for the case $\Lambda = 1$ and 0.5 this inequality, in the absence of blowing, is satisfied with an error not exceeding 25%. From inequality (23) it follows that the linearized equation is not applicable for describing the velocity field in the region of the boundary layer on a flat plate ($\Lambda = 0$).

For large blowing, as a consequence of formulas (12) and (14), the inequality (22) means that

$$4\Lambda + 2 \gg 2\Lambda - 1. \tag{24}$$

From this it is evident that with large blowing the error of the solution may be noticeable if Λ is not equal to 0.5.

2. The diffusion equation in the system of Eqs. (5) is analogous to the energy equation; therefore, in the sequel, we consider the diffusion equation only. The solution of this equation with the boundary conditions (6) has the form

$$C = B\int_0^{\eta} \exp\left(-\mathrm{Sc}\int_0^{\eta'} f(\eta)\,d\eta\right)d\eta',$$

where

$$\frac{1}{B} = \int_0^{\infty} \exp\left(-\mathrm{Sc}\int_0^{\eta'} f(\eta)\,d\eta\right)d\eta'. \tag{25}$$

In accord with Eqs. (25) and (8) the concentration profile increases monotonically from 0 to 1 as η increases and has an inflection point for $\eta = \eta^*$ ($f(\eta^*) = 0$), corresponding to a maximum value for $(dC/d\eta)$.

With moderate and with large blowing ($f_w > 1$) it may be assumed that $\eta^* \sim \lambda$. In this case the diffusion boundary layer is situated in a region including $\eta = \lambda$ with an effective thickness of order $\mathrm{Sc}^{-1/2}$.

For the case of small blowing ($f_w \ll 1$) the diffusion boundary layer is situated close to the surface. The effective thickness of the diffusion boundary layer turns out to be on the order of $\mathrm{Sc}^{-1/3}$.

If $\lambda\sqrt{\mathrm{Sc}} \gg 1$, $f_w \gg 1$, it is then obvious that in calculating the integral in the exponent of Eq. (25) in the region $\sqrt{\mathrm{Sc}}|\eta - \lambda| \leq 1$, we can replace the stream function (8) by $f = \eta - \lambda$, i.e., we can consider it in the potential flow approximation with the displacement thickness taken into account.

The following expression is obtained for the concentration distribution in this case:

$$C = B_1\{\Phi(\lambda\sqrt{\mathrm{Sc}}) - \Phi[(\lambda-\eta)\sqrt{\mathrm{Sc}}]\},$$

where

$$\frac{1}{B_1} = 1 + \Phi(\lambda\sqrt{\mathrm{Sc}}). \tag{26}$$

Formula (26) satisfactorily describes the concentration field in the maximum diffusional flow region ($\eta \approx \lambda$), but proves to be unsuitable for calculating the diffusional flow at the wall. To calculate the latter one can use formula (12).

Thus we obtain

$$\left(\frac{dC}{d\eta}\right)_w = 2B_1\sqrt{\frac{\mathrm{Sc}}{2\pi}}\exp\left\{-\mathrm{Sc}\, f_w^2\left[\frac{(2\Lambda+3)(2\Lambda+1)}{4\Lambda(2\Lambda+2)}\right]\right\}. \tag{27}$$

Formula (27) coincides with that obtained in [3] with wedge angle $\Lambda = 0.5$, and when $\Lambda = 1$, the exponent in Eq. (27) differs from that in [3] by 25%. The diffusional flow at the wall tends to zero asymptotically with an increase in blowing.

In the case of small Schmidt numbers ($\lambda\sqrt{\mathrm{Sc}} \ll 1$), the concentration field is also describable by formula (26), which enables us to calculate the diffusional flow:

$$\left(\frac{dC}{d\eta}\right)_w = 2B_1\sqrt{\frac{\mathrm{Sc}}{2\pi}}\exp\left(-\frac{\mathrm{Sc}\,\lambda^2}{2}\right). \tag{28}$$

For large Schmidt numbers and small blowing, formula (10) may be used throughout the diffusional boundary layer region. The relative concentration (25) has the form

$$C = B_2\int_0^{\eta}\exp\left[-\mathrm{Sc}\,\eta'\left(f_w + \frac{a\eta'^2}{6}\right)\right]d\eta',$$

where

$$\frac{1}{B_2} = \int_0^{\infty}\exp\left[-\mathrm{Sc}\,\eta'\left(f_w + \frac{a\eta'^2}{6}\right)\right]d\eta'. \tag{29}$$

In calculating the integral in formula (29) the main contribution to the exponent occurs in the domain $\eta \sim \mathrm{Sc}^{-1/3}$, therefore when the condition $(6/a)^{1/3} f_w \times \mathrm{Sc}^{2/3} \ll 1$ is satisfied, we can limit ourselves to the linear term in the expansion of $\exp(-\mathrm{Sc}\, f_w\eta)$.

Thus we obtain the following expression for the relative concentration

$$C = B_2\left[\gamma\left(\frac{1}{3}, \frac{\mathrm{Sc}\,a\eta^3}{6}\right) - (\mathrm{Sc}\, f_w)\left(\frac{\mathrm{Sc}\,a}{6}\right)^{-\frac{1}{3}}\gamma\left(\frac{2}{3}, \frac{\mathrm{Sc}\,a\eta^3}{6}\right)\right], \tag{30}$$

where

$$\frac{1}{B_2} = \left[\Gamma\left(\frac{1}{3}\right) - (\mathrm{Sc}\, f_w)\left(\frac{\mathrm{Sc}\,a}{6}\right)^{-\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)\right];$$

$$\gamma(\alpha, x) = \int_0^x \exp(-t)\, t^{\alpha-1}\, dt.$$

For $f_w \to 0$ formula (30) coincides with the expression obtained in [3].

Calculations were made using formula (26) with Schmidt number $\mathrm{Sc} = 0.7$ for the case $\Lambda = 1$ (Fig. 3).

The plotted points correspond to the numerical results obtained in [5] with the same blowing values. Deviation from the numerical results does not exceed 10-15%.

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